Minimum Sensitivity RC Active Driving-Point Immittance Synthesis

by

R. A. Rohrer

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ABSTRACT

General methods for RC-NIC realization of relatively arbitrary driving-point immittance functions are presented. A cascade-feedback structure is introduced to obtain minimum immittance pole sensitivity. Furthermore, techniques which may be frequently employed for minimum pole and zero sensitivity realization are discussed.

TABLE OF CONTENTS

I. INTRODUCTION

Although the field of RC active driving-point synthesis has received considerable attention in recent years, there still exists a need for straightforward realization techniques. An important consideration in this type of synthesis is the sensitivity of the immittance function to internal network changes. In this paper, methods of realization will be presented which not only produce a certain type of minimum sensitivity, but also yield semi-unique structures.

The significant step in any synthesis procedure is to decompose the given function into recognizable and realizable sub-functions. In RC active synthesis these functions must have alternating poles and zeros on the negative σ-axis. An optimum decomposition, yielding minimum sensitivity, has been found.[1,2]

Realizability conditions for arbitrary polynomials, given in terms of the numerator and denominator of the even or odd parts of positive real functions, have been derived.[2] These results have been applied to obtain structures and techniques for the actual RC active realization.[3]

These discussions bring us to the initial point of this report. A review of existing material is presented in the first three sections, with some new results added. The remaining sections would be most aptly integrated under the title, "Optimum Sensitivity Synthesis Techniques." Sections II and III cover the basic material requisite to RC active synthesis. Section IV gives an interpretation of the problem in terms of the well-known passive synthesis techniques. Section V gives the limitations on the functions which can be realized by the methods presented. Section VI introduces a structure useful in the minimum pole sensitivity synthesis of section VII. Section VIII covers a technique for achieving minimum pole and zero sensitivity.

II. THE IDEAL NIC

The two port active device commonly employed in RC active synthesis is the ideal negative impedance converter (NIC). The NIC is most conveniently defined in terms of the familiar g-parameters,[4] these being given as

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \begin{bmatrix} O & K \\ K & O \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} \qquad (1)$$

K is called the conversion ratio. The NIC so defined (see Fig. 1), which we shall call a K NIC, produces the input-output relations

$$Z_{11} = -\frac{1}{K^2} Z_{L2}$$

$$Z_{22} = -K^2 Z_{L1}$$

The definition (1) is equivalent to an ideal transformer of turns-ratio 1:K in cascade with a unity conversion (K=1) NIC (see Fig. 2a). As the reader may easily verify, a method of obtaining a K NIC (for $K > 1$) from an already existing unity conversion NIC is given in Fig. 2b. If K is to be less than one, the terminals, one and two, need merely be interchanged. The K NIC is not the most general (the non-diagonal terms of the g-matrix could be unequal), but it is particularly useful for the purposes of this report.

III. THE OPTIMUM DECOMPOSITION OF A POLYNOMIAL

Any polynomial in s, P(s), with real coefficients and no negative σ-axis roots may be factored into the difference of two, each having only negative σ-axis roots. The factorization depends on the transformation

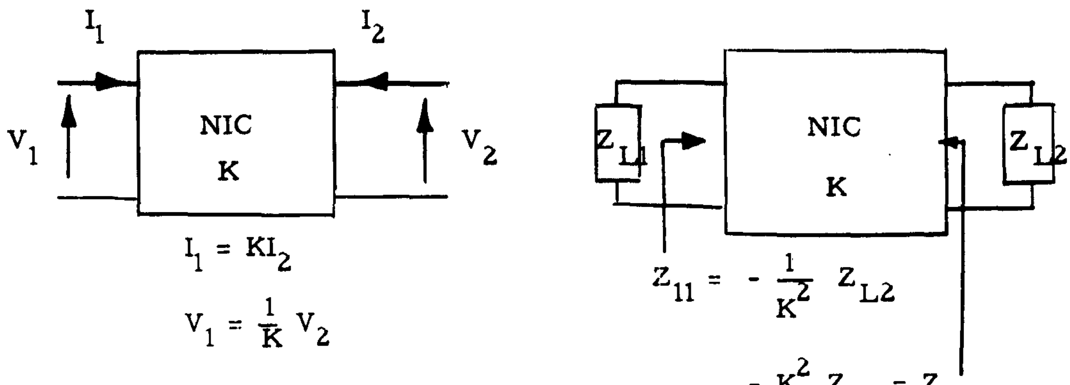
$$P(s) \longrightarrow P(s^2) \qquad (2)$$

The even polynomial, $P(s^2)$, may be factored into the unique product of a Hurwitz and an anti-Hurwitz polynomial

$$P(s^2) = \pm F(s)\, F(-s) \qquad (3)$$

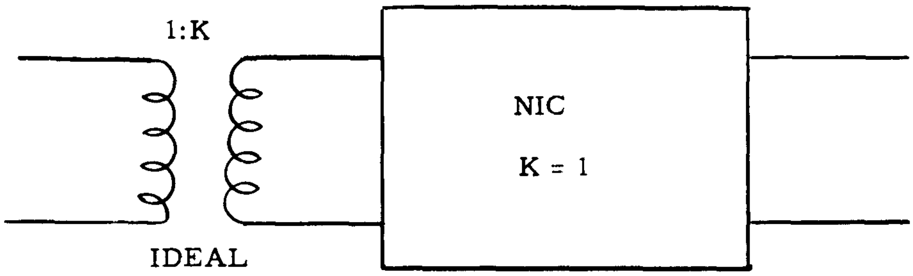
The positive sign emerges if the degree of P(s) is even, and the negative if the degree of P(s)* is odd.

* "The degree of P(s)" will be henceforth defined by $P(s)^o$.

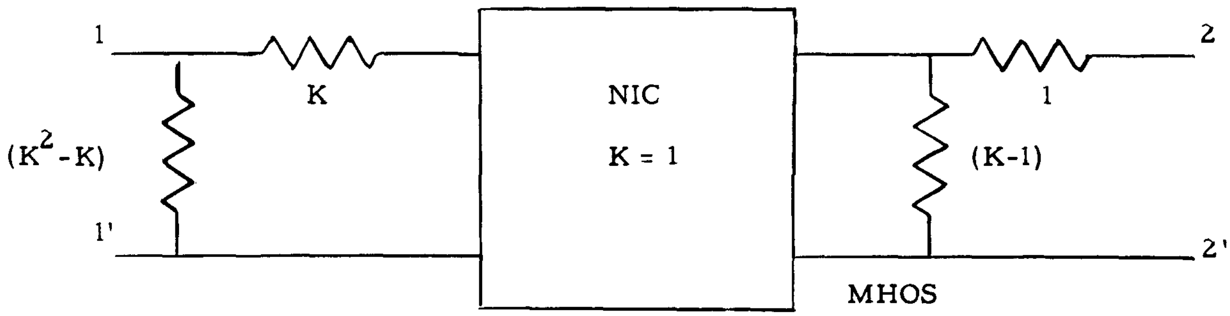


a) NIC Defined b) NIC Functions

FIG. 1--THE IDEAL NIC AND ITS FUNCTIONS



a) K NIC Equivalent Circuit



b) Obtaining K NIC From Unity NIC

FIG. 2--THE K NIC

This factoring process may be represented pictorially, as in Fig. 3, which shows the identification of LHP roots of $P(s^2)$ with $F(s)$ and RHP

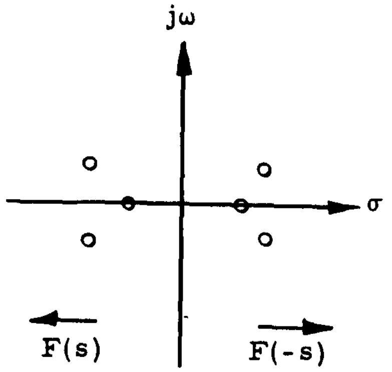


FIG. 3--s-PLANE FACTORING OF $P(s^2) = F(s) \cdot F(-s)$.

roots with $F(-s)$. The s-plane representation of the factorization gives an intuitive explanation for why we cannot have negative σ-axis roots. $P(s^2)$, being an even polynomial, exhibits quadrantal symmetry in the s-plane.[5] Negative σ-axis roots of $P(s)$ are of the form $(s + \sigma_o)$, which upon transformation (2) becomes $(s^2 + \sigma_o)$. It is clear from its s-plane representation that this term cannot be the product of LHP and RHP factors (see Fig. 4).

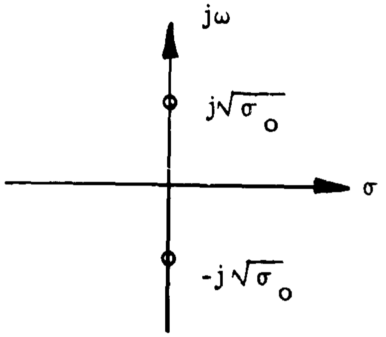


FIG. 4--s-PLANE ROOTS OF $(s^2 + \sigma_o)$.

The reader might conjecture that a double-order negative σ-axis root could be thus factored. However, just as there are residue conditions on the jω-axis poles of positive real [p. r.] functions, negative σ-axis roots have similar restrictions in RC active synthesis. Methods to handle negative

σ-axis roots will be discussed in a later section.

Returning to the decomposition of $P(s^2)$, we note that the Hurwitz and anti-Hurwitz polynomials may be expressed in terms of their even and odd parts

$$F(s) = m(s) + n(s)$$
$$F(-s) = m(s) - n(s)$$

Therefore, from Eq. (3)

$$P(s^2) = \pm [m^2(s) - n^2(s)] \tag{4}$$

The even and odd nature of m(s) and n(s) allows them to be identified as

$$m(s) = a(s^2)$$
$$n(s) = sb(s^2)$$

Substitution of these relations into Eq. (4) yields

$$P(s^2) = \pm [a^2(s^2) - s^2b^2(s^2)] \tag{5}$$

Finally, we must use the inverse of the original transformation (2)

$$P(s^2) \longrightarrow P(s) \tag{6}$$

This gives

$$P(s) = \pm [a^2(s) - sb^2(s)] \tag{7}$$

Both a(s)/sb(s) and b(s)/a(s) are guaranteed to be RC driving point impedances because m(s)/n(s) and n(s)/m(s) are LC driving point impedances.[6]

It has been shown that the decomposition (7) yields minimum sensitivity of every simple root of P(s) with respect to the gain* of the NIC ultimately used.[2] It also gives minimum coefficient of P(s) sensitivity.[1] These optimum conditions apply regardless of the synthesis procedure used; hence, Eq. (7) is called the "optimum decomposition."

* The NIC gain here is the product $g_{12}g_{21} = K^2$.

IV. RC-LC TRANSFORMATION AND ODD PART IMMITTANCE SYNTHESIS

The synthesis problem is related to the successful decomposition of a ratio of two polynomials (a rational immittance function); hence, we must relate the optimum decomposition of the preceding section to the decomposition of a polynomial ratio. This can be best done by applying a transformation similar to (2) to an immittance function and investigating the resulting function.

A transformation similar to (2), which furthermore embodies a physical significance, can be found from the RC-LC network equivalence:

$$Z\Big|_{RC} = \frac{1}{s}\,\frac{P(s)}{Q(s)} \Longrightarrow Z\Big|_{LC} = \frac{1}{s}\,\frac{P(s^2)}{Q(s^2)} \tag{8a}$$

$$Y\Big|_{RC} = s\,\frac{P(s)}{Q(s)} \Longrightarrow Y\Big|_{LC} = s\,\frac{P(s^2)}{Q(s^2)} \tag{8b}$$

These relationships are those which emerge when each of the R's of an RC network is replaced by L's of equal value to form an analogous LC network.[7] The RC-LC transformations is the mathematical expression of the equivalence (8) and is given by

$$Z(s) \longrightarrow sZ(s^2) \tag{9a}$$
$$Y(s) \longrightarrow \frac{1}{s}\,Y(s^2) \tag{9b}$$

Consider an admittance function

$$Y(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{10}$$

The odd part of this admittance is given by

$$\text{OD } Y(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2} \tag{11}$$

We may note the similarity of the denominator in Eq. (11) to the decomposition, Eq. (4). The identification which facilitates the RC active

synthesis is to equate the transformed functions (9) with the odd parts of positive real functions:

$$s\,Z(s^2) = \mathrm{OD}\,Z'(s) \tag{12a}$$

$$\frac{1}{s}\,Y(s^2) = \mathrm{OD}\,Y'(s) \tag{12b}$$

An RC immittance synthesis with optimum pole sensitivity could be accomplished if the odd part could be realized in an LC structure, for changing the L's to R's of equal value would be equivalent to taking the inverse transformation (6).

To obtain an LC realization for OD Y(s), we must consider the input admittance of the cascade structure* of Fig. 5, which is

$$Y_{11} = y_{11}\,\frac{\dfrac{1}{z_{22}} - Y_L}{y_{22} - Y_L} \tag{13}$$

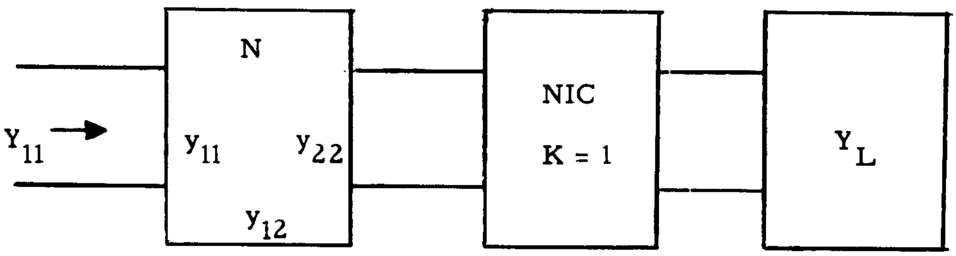


FIG. 5--CASCADE STRUCTURE FOR REALIZATION OF OD Y(s).

Eq. (11) may be factored to obtain

$$\mathrm{OD}\,Y(s) = \frac{m_1}{n_2}\,\frac{\dfrac{n_1}{m_1} - \dfrac{n_2}{m_2}}{\dfrac{m_2}{n_2} - \dfrac{n_2}{m_2}} \tag{14}$$

---

* Mitra and Kuh (ref. 3) used this structure from the RC standpoint.

which suggests the identifications

$$y_{11} = \frac{m_1}{n_2}, \quad y_{22} = \frac{m_2}{n_2}, \quad \frac{1}{z_{22}} = \frac{n_1}{m_1}, \quad Y_L = \frac{n_2}{m_2}; \quad \text{Case A} \qquad (15)$$

Or, Eq. (11) may be factored in a different manner to obtain

$$\text{OD } Y(s) = \frac{n_1}{m_2} \; \frac{\dfrac{m_1}{n_1} - \dfrac{m_2}{n_2}}{\dfrac{n_2}{m_2} - \dfrac{m_2}{n_2}} \qquad (16)$$

which suggests the identifications

$$y_{11} = \frac{n_1}{m_2}, \quad y_{22} = \frac{n_2}{m_2}, \quad \frac{1}{z_{22}} = \frac{m_1}{n_1}, \quad Y_L = \frac{m_2}{n_2}; \quad \text{Case B} \qquad (17)$$

The network, N, is in both cases recognized to be the lossless network realized in the Darlington method of synthesizing a driving-point admittance in terms of a lossless network terminated in a one mho conductance.[8] Furthermore, the load admittance is an LC driving-point function. It follows that in Case A

$$y_{12} = \frac{\sqrt{m_1 m_2 - n_1 n_2}}{n_2} \qquad (18)$$

In Case B

$$y_{12} = \frac{\sqrt{n_1 n_2 - m_1 m_2}}{m_2} \qquad (19)$$

As in the Darlington synthesis, Case A and Case B are mutually exclusive. Moreover, the usual difficulty of making y_{12}^2 a perfect square applies. However, we are awarded an additional degree of freedom because in starting from the odd part, the even part is not uniquely determined. Hence, any positive constant may be added to the even part, which will quite often allow us to make y_{12}^2 a perfect square without resorting to augmentation.

The Darlington realization is always possible, therefore, this realization of OD Y(s) is always possible. To obtain the RC admittance function which precipitated the problem, one needs merely to change all of the L's to R's of the same value.

These remarks apply equally well on the dual basis as the following example will illustrate.

Example: Let

$$Z(s) = \frac{36s^2 + 26s + 2}{36s^2 + 8s + 1} \tag{20}$$

This, upon transformation (12a), yields

$$\text{OD } Z'(s) = \frac{36s^5 + 26s^3 + 2s}{36s^4 + 8s^2 + 1}$$

The impedance, constructed from its odd part, is

$$Z'(s) = \frac{6s^3 + 2s^2 + 4s + 1}{6s^2 + 2s + 1}$$

As in the Darlington synthesis, the next step is to construct the even part

$$\text{EV } Z'(s) = \frac{1}{36s^4 + 8s^2 + 1}$$

Hence, we must use Case A and make the dual identifications of those in Eqs. (14) and (17)

$$z_{11} = \frac{2s^2 + 1}{2s} = s + \frac{1}{2s}$$

$$z_{22} = \frac{6s^2 + 1}{2s} = 3s + \frac{1}{2s}$$

$$z_{12} = \frac{1}{2s}$$

$$Z_L = \frac{2s}{6s^2 + 1}$$

These may be used to obtain the LC realization of Fig. 6. Changing the L's to R's of Equal value yields the RC realization in Fig. 7 of the original function, Eq. (20).

Not all problems can be worked out as easily as this one. The Darlington synthesis can and frequently does depend on ideal transformers in the lossless network. Mitra and Kuh have suggested an excellent method

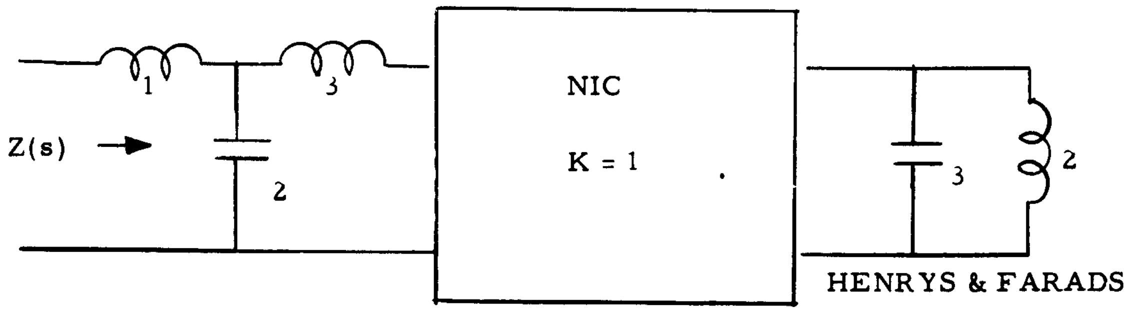


FIG. 6--LC REALIZATION OF $Z(s) = \dfrac{36s^5 + 26s^3 + 2s}{36s^4 + 8s^2 + 1}$

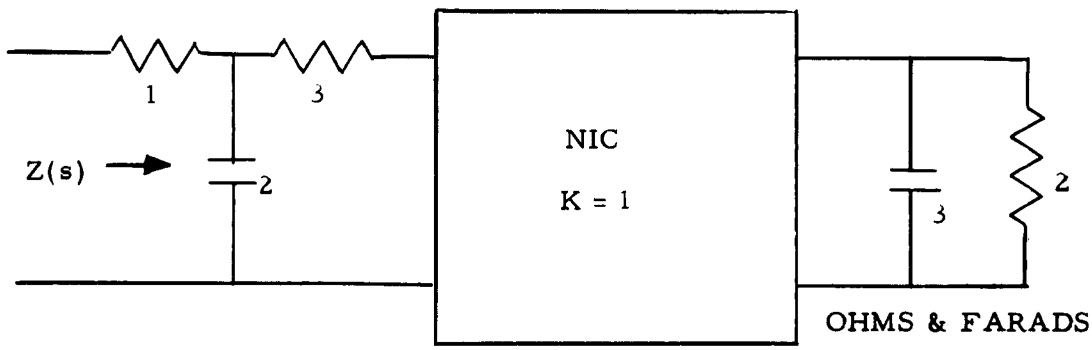


FIG. 7--RC REALIZATION OF $Z(s) = \dfrac{36s^2 + 26s + 2}{36s^2 + 8s + 1}$

for eliminating ideal transformers from the cascade structure.[3] Once one has obtained the network parameters, Eq. (15) or (17) and (18) or (25), he may realize y_{11} satisfying the zeros of y_{12} (by the zero-shifting and removing technique or, at worst, as a balanced lattice). This procedure will actually realize a network with parameters

$$y_{11},\ H\,y_{12},\ \text{and},\ y'_{22} \tag{21}$$

Examination of Eq. (13) reveals that it may also be written as

$$Y_{11} = y_{11} - \frac{y_{12}^2}{y_{22} - Y_L} \tag{22}$$

Furthermore, this is entirely equivalent to

$$Y_{11} = y_{11} - \frac{(H\,y_{12})^2}{H^2 y_{22} - H^2 Y_L} \tag{23}$$

Hence, the admittance Eq. (13) may also be obtained from the network Eq. (21) with its output load given by

$$Y'_L = H^2 y_{22} - y'_{22} - H^2 Y_L \tag{24}$$

Fig. 8 shows the implementation of this. Aside from $H^2 Y_L$, which must be entirely negative, the remaining portion of Y'_L, $H^2 y_{22} - y'_{22}$, may be a sum of positive and negative admittances

$$H^2 y_{22} - y'_{22} = Y^+ - Y^- \tag{25}$$

These admittances may be placed on appropriate sides of the NIC for the final realization of Fig. 9.

We now have at our disposal fairly general methods for minimum pole sensitivity synthesis. However, these are limited by some minor practical drawbacks. The most serious of these is that one may not always obtain a common-ground structure. A structure which solves

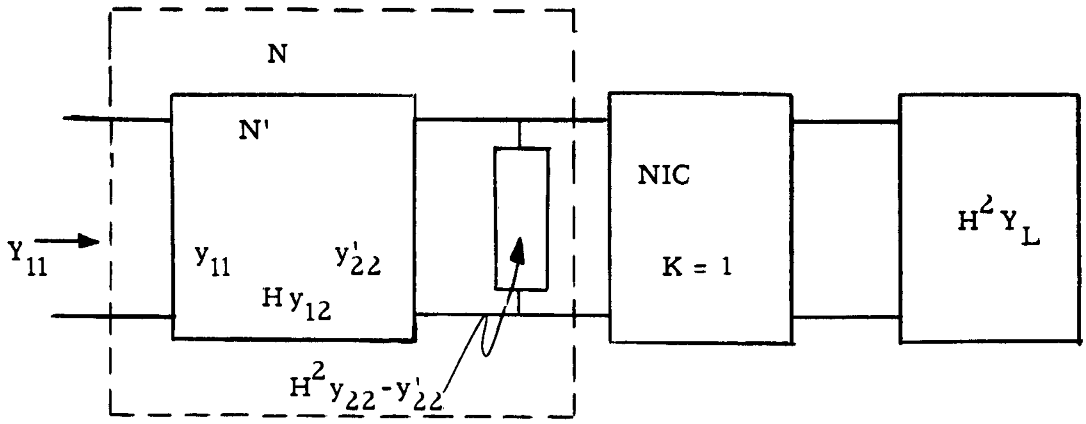


FIG. 8--TRANSFORMERLESS REALIZATION OF EQ. (23)

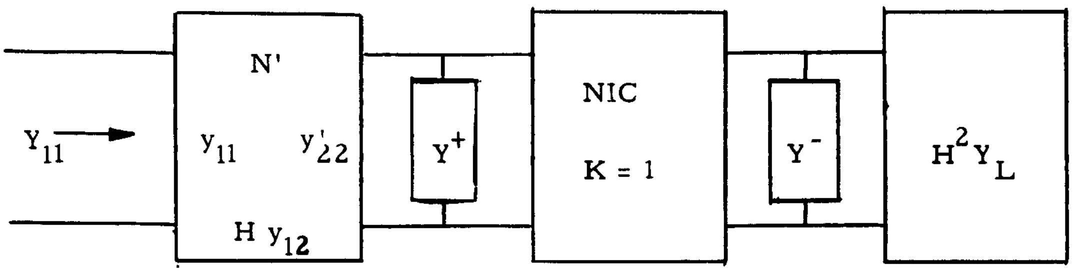


FIG. 9--TRANSFORMERLESS REALIZATION OF EQS. (23) AND (25)

this problem and, furthermore, requires no augmentation will be discussed in section VI.

V. REALIZABILITY CONDITIONS

The discussion of the preceding section revealed a method of obtaining an RC realization of an immittance function when its transform (9) can be identified with an immittance odd part. The odd part identification, Eq. (12), restricts the transformed functions, and, therefore, restricts, the original function as well. To obtain the realizability

conditions which limit this technique, we must study the properties of the odd parts of positive real immittances.

The degree of the numerator of a p.r. immittance can at most be one greater than that of the denominator.[9] Consider the immittance

$$G(s) = \frac{m_1 + n_1}{m_2 + n_2}$$

Its odd part is given as

$$\text{OD } G(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

Since, at most,

$$n_1^o = m_2^o + 1$$

or

$$m_1^o = n_2^o + 1,$$

the degree of the numerator of OD G(s) can be no more than one greater than that of the denominator. In other words, if

$$\text{OD } G(s) = \frac{N(s)}{D(s)} \tag{26a}$$

then

$$N(s)^o \leq D(s)^o + 1 \tag{26b}$$

Consider

$$Z(s) = \frac{P(s)}{Q(s)}$$

Eq. (12a) gives

$$\text{OD } Z'(s) = s\,\frac{P(s^2)}{Q(s^2)}$$

Hence, Eq. (26b) yields

$$(s\ P(s^2))^o \leq Q(s^2)^o + 1$$

Therefore,

$$P(s^2)^o \leq Q(s^2)^o$$

which is equivalent to

$$P(s)^o \leq Q(s)^o \tag{27}$$

Similarly, consider

$$Y(s) = \frac{A(s)}{B(s)}$$

Eq. (12b) gives

$$\text{OD } Y'(s) = \frac{1}{s} \frac{A(s^2)}{B(s^2)}$$

Hence, Eq. (26b) yields

$$A(s^2)^o \leq (sB(s^2))^o + 1$$

Therefore,

$$A(s^2)^o \leq B(s^2)^o + 2$$

which is equivalent to

$$A(s)^o \leq B(s)^o + 1 \tag{28}$$

Aside from the degree restrictions, the only other consideration is that of negative σ-axis poles. In the transformed function, Eqs. (12), these become $j\omega$-axis poles, and they are particularly important because OD $Z'(s)$ always has a pole introduced at infinity and OD $Y'(s)$ always has a pole introduced at the origin (see Eqs. (12)).

Any p. r. immittance may be expressed as a sum of a number of purely reactive (susceptive) terms and a single minimum reactive (susceptive) term

$$G(s) = \frac{K_o}{s} + K_\infty s + \sum_i \frac{K_i s}{s^2 + \omega_i^2} + G_m(s)$$

G_m denotes a minimum reactive (susceptive) immittance. The residue condition for p. r. functions requires that

K_i real and $K_i \geq 0, \quad \forall i$

The odd part of the immittance is given by

$$\text{OD } G(s) = \frac{K_o}{s} + K_\infty s + \sum_i \frac{K_i s}{s^2 + \omega_i^2} + \text{OD } G_m(s)$$

This follows from the linearity of the operation of taking the odd part. From the foregoing discussion it is clear that OD $G_m(s)$ will have no $j\omega$-axis poles. Hence, all $j\omega$-axis poles in OD G(s) must have real and positive residues. Therefore, negative σ-axis poles of a given impedance must have positive residues, and negative σ-axis poles of a given admittance divided by s must have positive residues.

The residue condition is more succinctly stated in terms of the transforms (9), as all $j\omega$-axis poles of the transformed function must have real and positive residues. The residue condition in the pole at infinity of OD Z'(s) and that at the origin of OD Y'(s) implies that

$$Z(\infty) \geq 0 \tag{29a}$$

$$Y(0) \geq 0 \tag{29b}$$

These restrictions completely define the functions which we may realize by the methods of this paper. The realizability conditions are repeated here for completeness:

The restrictions on $Z(s) = N(s)/D(s)$ are

i) $N(s)^o \leq D(s)^o$, (30a)

ii) Negative σ-axis poles have real and positive residues, (30b)

iii) $Z(\infty) \geq 0$. (30c)

The restrictions on $Y(s) = A(s)/B(s)$ are,

i) $N(s)^o \leq D(s)^o + 1$, (31a)

ii) Negative σ-axis poles of $Y(s)/s$ have real and positive residues, (31b)

iii) $Y(0) \geq 0$. (31c)

The most effective method of dealing with the negative σ-axis poles is to remove them after the transformation (9), as $j\omega$-axis poles, in a manner analogous to the Brune preamble.[10] Realizable functions which have no negative σ-axis poles shall be called "minimum active."

It must be emphasized that the restrictions, (30 and 31), apply only for the synthesis discussed. Sipress has shown that any rational function can be realized with R's, C's and one NIC.[11]

VI. THE CASCADE-FEEDBACK STRUCTURE

The most general one NIC structure, which may be used for the realization of a driving point immittance, consists of a passive, reciprocal, 3-port with an NIC imbedded (see Fig. 10).

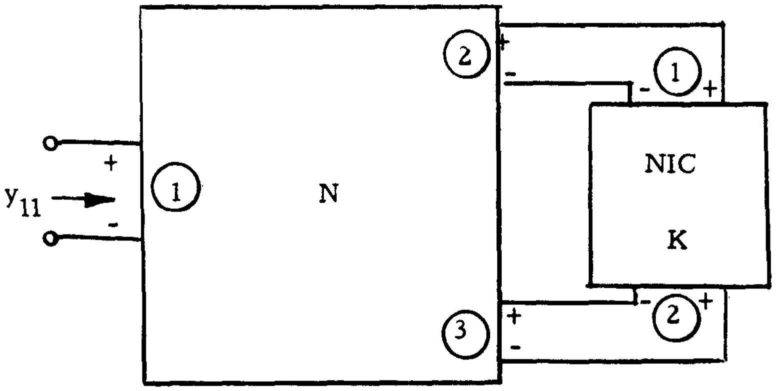


FIG. 10--MOST GENERAL SINGLE NIC DRIVING POINT IMMITTANCE REALIZATION

Consider the parameters of the passive network, N, to be specified on the admittance basis

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & y_{23} \\ y_{13} & y_{23} & y_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{32}$$

The NIC constraints, Eq. (1), when applied to Eq. (32) yield

$$Y_{11} = \frac{I_1}{E_1} = y_{11} - \frac{y_{12}^2 - K^2 y_{13}^2}{y_{22} - K^2 y_{33}} \tag{33}$$

From this we may see that the value of y_{23} is immaterial, and may, for convenience, be set equal to zero. A simplification results from making the identifications

$$\begin{aligned} y_{11} &= y_{11a} + y_{11b} \\ y_{12} &= y_{12a} \\ y_{13} &= y_{12b} \\ y_{22} &= y_{22a} \\ y_{33} &= y_{22b} \end{aligned} \tag{34}$$

The relations (34), when inserted into Eq. (33) give

$$Y_{11} = y_{11a} - \frac{y_{12a}^2}{y_{22a} - K^2 y_{22b}} + y_{11b} - \frac{K^2 y_{12b}^2}{K^2 y_{22b} - y_{22a}} \tag{35}$$

This is readily seen to be the cascade output, through an NIC, of two parallel input, two-port structures, as is pictured in Fig. 11.

Duality may be used throughout this development and the dual structure of Fig. 12 is specified by

$$Z_{11} = \frac{E_1}{I_1} = z_{11a} - \frac{K^2 z_{12a}^2}{K^2 z_{22a} - z_{22b}} + z_{11b} - \frac{z_{12b}^2}{z_{22b} - K^2 z_{22a}} \tag{36}$$

The structures introduced here, although not the most general because the NIC used is not the most general, will prove quite powerful in the solution of minimum sensitivity synthesis problems.

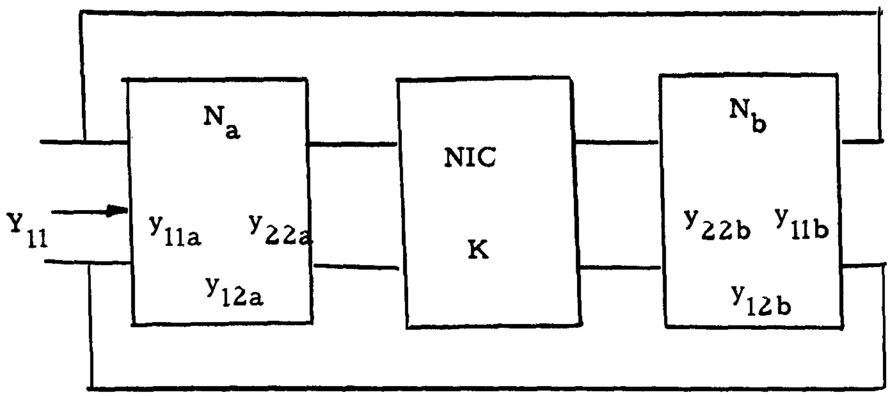


FIG. 11--CASCADE-FEEDBACK IMPLEMENTATION OF EQ. (35).

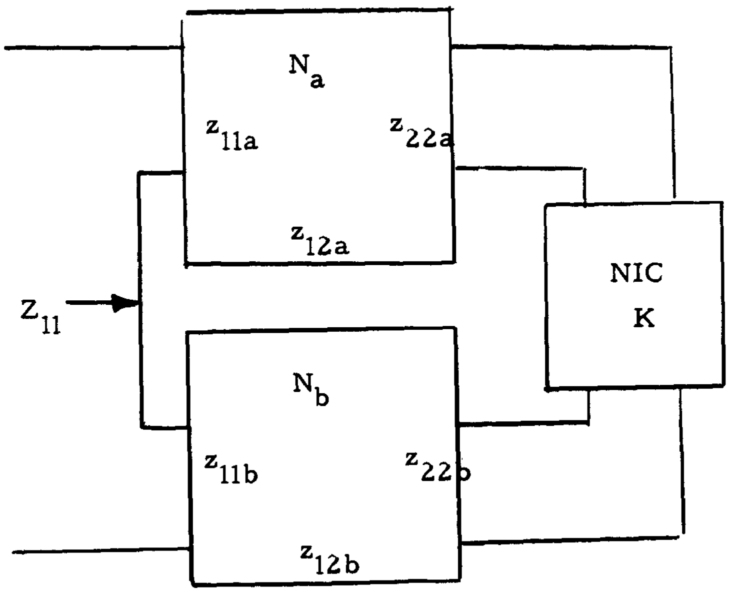


FIG. 12--CASCADE-FEEDBACK IMPLEMENTATION OF EQ. (36).

VII. OPTIMUM POLE SENSITIVITY SYNTHESIS TECHNIQUE

A minimum active admittance, which satisfies the realizability conditions (31), may be related to the odd part of a p. r. admittance through the transformation (9b). The odd part of the admittance may be used to compute the even part

$$\mathrm{EV}\ Y'(s) = \frac{N(s^2)}{m_2^2 - n_2^2}$$

The numerator of the even part exhibits quadrantal symmetry in the s-plane. Therefore, it may be decomposed in the manner of Eq. (4), yielding

$$\mathrm{EV}\ Y'(s) = \frac{m_1^2 - n_1^2}{m_2^2 - n_2^2}$$

The right hand side of this equation can be represented as the sum of two even part components, i. e.,

$$\mathrm{EV}\ Y'(s) = \frac{m_1^2}{m_2^2 - n_2^2} + \frac{-n_1^2}{m_2^2 - n_2^2}$$

Each of the terms on the right hand side of this equation may be identified as the even part of a p. r. admittance

$$\mathrm{EV}\ Y_a(s) = \frac{m_1^2}{m_2^2 - n_2^2}$$

$$\mathrm{EV}\ Y_b(s) = \frac{-n_1^2}{m_2^2 - n_2^2}$$

These even parts may each be used to generate a minimum susceptive admittance

$$Y_a(s) = \frac{m_1' + n_1'}{m_2 + n_2}$$

$$Y_b(s) = \frac{m_1'' + n_1''}{m_2 + n_2}$$

The parameters which realize the pertinent odd parts (see Eqs. (15), (17), (18), and (19)) may be found by inspection to be

$$Y_a: \quad y_{11a} = \frac{m_1'}{n_2}, \quad y_{12a} = \frac{m_1}{n_2}$$
$$y_{22a} = \frac{m_2}{n_2}, \quad Y_{La} = \frac{n_2}{m_2} \tag{37a}$$

$$Y_b: \quad y_{11b} = \frac{n_1''}{m_2}, \quad y_{12b} = \frac{n_1}{m_2}$$
$$y_{22b} = \frac{n_2}{m_2}, \quad Y_{Lb} = \frac{m_2}{n_2} \tag{37b}$$

These networks are, as expected, loaded by each other, i. e.

$$y_{22a} = Y_{Lb}, \; y_{22b} = Y_{La}$$

Hence, the cascade-feedback configuration of Fig. 11 is possible.

This procedure has allowed us always to obtain y_{12}^2 's which are perfect squares, thus ending any argument as to the necessity of augmentation. Moreover, since m and n are even and odd parts respectively of a Hurwitz polynomial, their zeros are all on the $j\omega$-axis, and the networks are always amenable to a ladder (common-ground) realization.

Example: Let

$$Y(s) = \frac{30s^2 + 282s}{s^2 - 5s + 4} \tag{38}$$

(Note: $Y(0) = 0$, satisfying condition (31c).)

The transformation (9b) yields

$$\text{OD } Y'(s) = \frac{30s^3 + 282s}{s^4 - 5s^2 + 4}$$

From this we may obtain the p. r. function

$$Y'(s) = \frac{s^2 + 33s + 72}{s^2 + 3s + 2}$$

Its even part is

$$\text{EV } Y'(s) = \frac{s^4 - 25s^2 + 144}{s^4 - 5s^2 + 4}$$

The numerator here is not a perfect square; however, we may use the decomposition, Eq. (4), to express it as the difference of two squares, i. e. ,

$$m_1^2 - n_1^2 = s^4 - 25s^2 + 144 = (s^2 + 12)^2 - (7s)^2$$

Therefore, the even part may be expressed as

$$\text{EV } Y'(s) = \frac{(s^2+12)^2}{s^4 - 5s^2 + 4} + \frac{-49s^2}{s^4 - 5s^2 + 4}$$

Hence, we have the identifications

$$\text{EV } Y_a = \frac{(s^2 + 12)^2}{s^4 - 5s^2 + 4}$$

$$\text{EV } Y_b = \frac{-49s^2}{s^4 - 5s^2 + 4}$$

These even parts are used to generate their respective minimum suspectance p. r. admittance functions

$$Y_a = \frac{s^2 + \frac{50}{3} s + 72}{s^2 + 3s + 2}$$

$$Y_b = \frac{49/3\ s}{s^2 + 3s + 2}$$

At this point a convenient check is to see that the sum of Y_a and Y_b is equal to Y'. The network parameters may now be identified, according to Eqs. (37), as

$$y_{11a} = \frac{s^2 + 72}{3s} = \frac{s}{3} + \frac{24}{s}$$

$$y_{12a} = -\frac{(s^2 + 12)}{3s} = -\frac{s}{3} - \frac{4}{5}$$

$$y_{22a} = Y_{Lb} = \frac{s^2+2}{3s} = \frac{s}{3} + \frac{2}{3s}$$

$$y_{11b} = \frac{49/3\ s}{s^2 + 2}$$

$$y_{12b} = \frac{-7s}{s^2 + 2}$$

$$y_{22b} = Y_{La} = \frac{3s}{s^2 + 2}$$

Fig. 13 gives the LC realization indicated here, with the negative elements through the NIC, we obtain the final realization (Fig. 14) for Eq. (38).

In general the negative elements do not appear adjacent to the NIC in the ladder realization of the separate networks. However, the method of transformerless synthesis discussed in section IV, coupled with the use of the NIC conversion ratio, K, can always surmount these difficulties.

Suppose that in attempting to eliminate transformers from the networks of Fig. 11, the two associated constants, H_a and H_b, are unequal. The NIC turns-ratio may then be adjusted such that

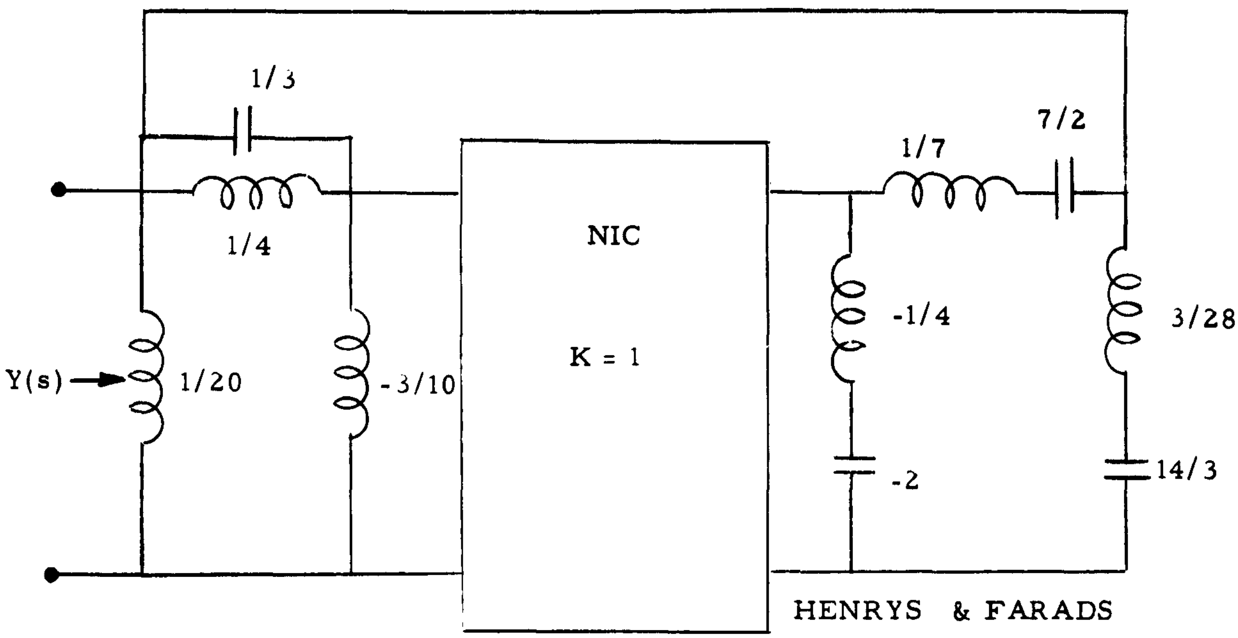


FIG. 13--LC REALIZATION OF $Y(s) = \dfrac{30s^3 + 282s}{s^4 - 5s^2 + 4}$

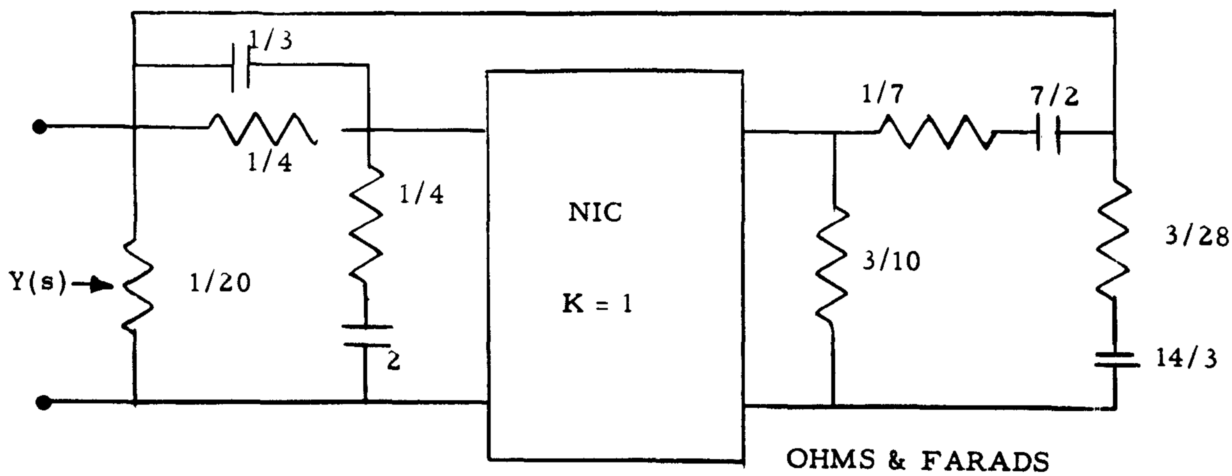


FIG. 14--REALIZATION OF $Y(s) = \dfrac{30s^2 + 282s}{s^2 - 5s + 4}$

$$K = \frac{H_a^2}{H_b^2} \tag{39}$$

It is easily seen that for this value of K both networks satisfy Eq. (29).

For an example, let us reconsider the same problem, Eq. (38), which we have just solved for a unity conversion NIC. The zero shifting and removing technique[12] will realize y_{11a} satisfying the zeros of y_{12a}, such that

$$y_{11a} = \frac{s^2 + 72}{3s}$$

$$y_{12a} = \frac{s^2 + 12}{3s}$$

$$y'_{22a} = \frac{s^2 + 12}{3s}$$

Comparing these to the previous set, we see that

$$H_a = 1$$

Furthermore,

$$H_a^2 \; y_{22a} - y'_{22a} = -\frac{10}{3s}$$

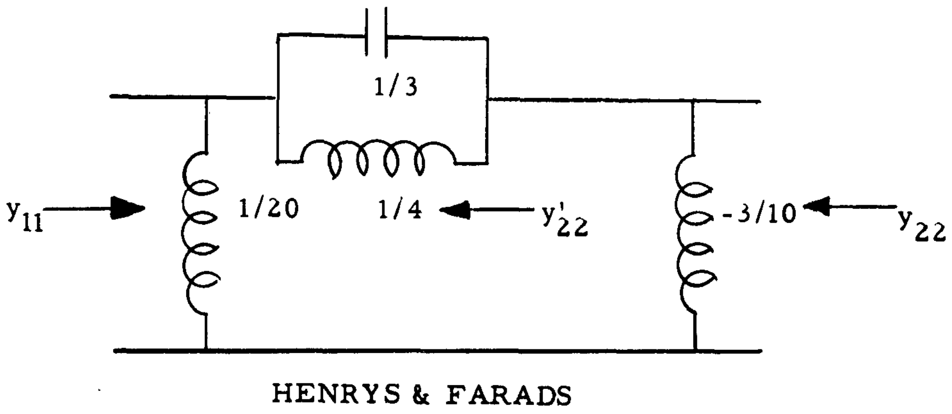


FIG. 15--TRANSFORMERLESS REALIZATION OF N_a

These parameters define the network of Fig. 15, which is seen to be the same as N_a of Fig. 13. The same technique applied to the parameters of Y_b yields

$$y_{11b} = \frac{49/3\ s}{s^2 + 2}$$

$$y_{12b} = - \frac{49/3\ s}{s^2 + 2}$$

$$y'_{22b} = \frac{49/3\ s}{s^2 + 2}$$

This realization, shown in Fig. 16, gives

$$H_b = \frac{7}{3}$$

Furthermore,

$$H_b^2\, y_{22b} - y'_{22b} = 0$$

The LC realization of Fig. 17 is made by combining the two networks with an NIC of conversion ratio given by Eq. (39) as

$$K = \frac{9}{49}$$

The final realization obtained by changing all of the L's to R's of the same value is given in Fig. 18.

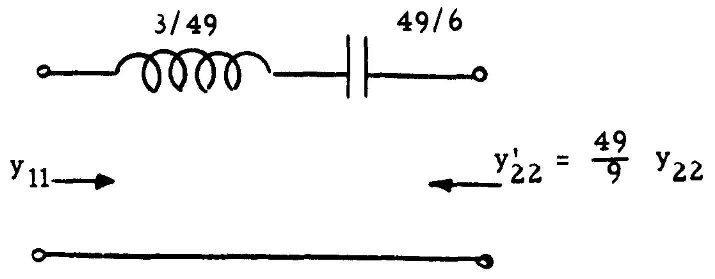


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FIG. 16--TRANSFORMERLESS REALIZATION OF N_b

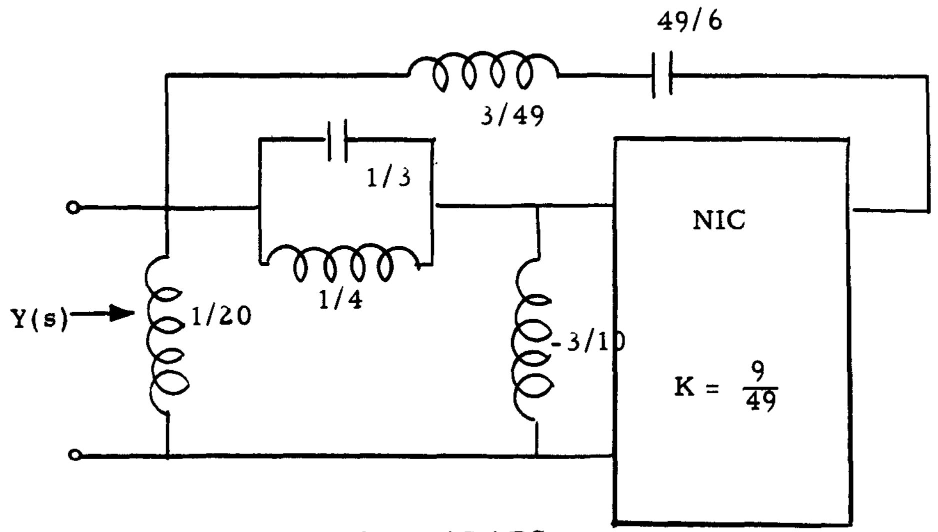

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FIG. 17-- ALTERNATE LC REALIZATION OF $Y(s) = \dfrac{30s^3+282s}{s^4-5s^2+4}$

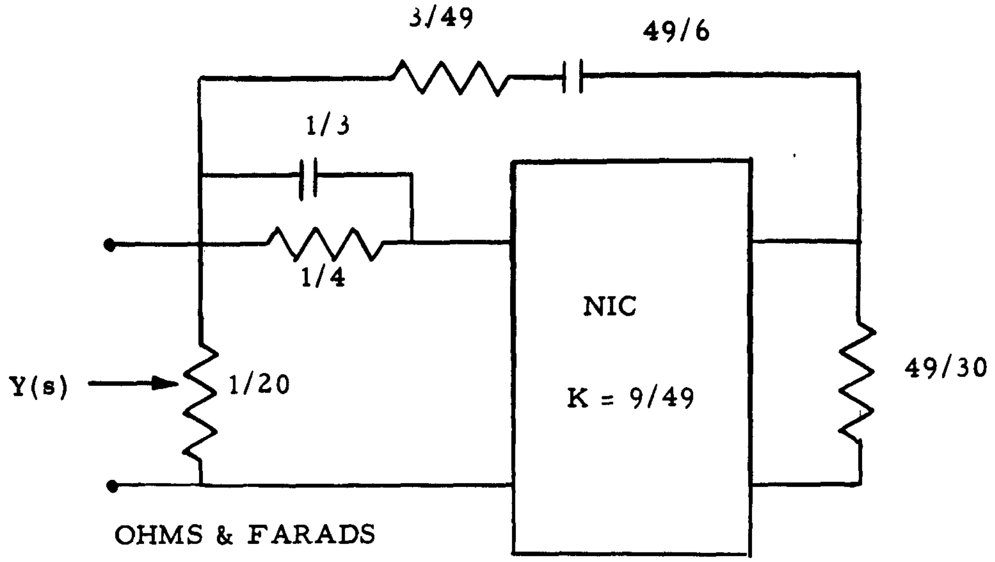


FIG. 18--ALTERNATE REALIZATION OF $Y(s) = \dfrac{30s^2 + 282s}{s^2 - 5s + 4}$

VIII. MINIMUM SENSITIVITY IMMITTANCE SYNTHESIS TECHNIQUE

The root and coefficient sensitivities of a polynomial are dependent only upon the decomposition used, and not upon the subsequent method of synthesis. Hence, if both the numerator and the denominator can be decomposed in the manner of Eq. (7), an optimum sensitivity synthesis may result. In the previous sections methods dealt only with minimum sensitivity denominator decomposition; in this section we shall attempt to extend those techniques to the numerator decomposition as well.

Since the decomposition, Eq. (4) is to be applied to the numerator, a further restriction must be added to (30) and (31). When the minimum active function has been obtained, its reciprocal must also be minimum active (with odd part transformed functions, we must, of course, overlook the zero at the origin, which always appears in the odd part). In other words, we may only handle functions with no poles or zeros on the negative σ-axis.

The procedure for minimum sensitivity immittance synthesis is quite similar to that of section VII. The difference here is that both numerator and denominator are immediately decomposed, whereas in the former case the denominator decomposition was not undertaken until the corresponding even part was obtained. On the impedance basis, after the transformation (9a), we have from Eq. (12a)

$$\mathrm{OD}\ Z'(s) = s\,Z(s^2) = s\,\frac{m_1^2 - n_1^2}{m_2^2 - n_2^2} \tag{40}$$

That this decomposition can always be effected follows from the discussion above, disallowing negative σ-axis poles or zeros of Z(s). The right hand side of Eq. (40) can be factored into the sum of two odd parts

$$\mathrm{OD}\ Z'(s) = \frac{s\,m_1^2}{m_2^2 - n_2^2} + \frac{-s\,n_1^2}{m_2^2 - n_2^2} \tag{41}$$

Each of these terms can be identified separately as the odd part of a p. r. function

$$\mathrm{OD}\ Z_a = \frac{s\,m_1^2}{m_2^2 - n_2^2} \tag{42a}$$

$$\mathrm{OD}\ Z_b = \frac{-s\,n_1^2}{m_2^2 - n_2^2} \tag{42b}$$

The even parts corresponding to these may be generated such that their numerators are perfect squares. This follows because there is an arbitrary constant which may be added to the even part when it is determined from the odd part; or, the odd parts of the given of a given function may always be augmented. The constant guarantees the existence of one more unknown than the number of equations. Therefore, the final equation may be that which constrains the numerator to be a perfect square.

The even parts, then, are

$$\mathrm{EV}\ Z_a = \frac{m_1'm_2 - n_1'n_2}{m_2^2 - n_2^2} \quad : \quad \underset{\text{Rational}}{\sqrt{\pm(m_1'm_2 - n_1'n_2}} \tag{43a}$$

$$\mathrm{EV}\ Z_b = \frac{m_1''m_2 - n_1''n_2}{m_2^2 - n_2^2} : \underset{\text{Rational}}{\sqrt{\pm(m_1''m_2 - n_1''n_2)}} \tag{43b}$$

The reader may note that a further restriction is that the two even parts must be chosen such that the functions Z_a and Z_b are of opposite case, Eqs. (15) and (17). Once we have attained this point, the synthesis is straight-forward and follows the development of section VII from Eq. (37). An example will amplify the above presentation of the synthesis procedure.

Example: Let

$$Z(s) = \frac{1 - s}{4 - s} \tag{45}$$

Upon applying the transformation (9a), we obtain from Eq. (12a)

$$\mathrm{OD}\ Z'(s) = s\,\frac{1 - s^2}{4 - s^2}$$

This may be factored in the manner of Eq. (41) as

$$\mathrm{OD}\ Z'(s) = \frac{s}{4 - s^2} + \frac{-s^3}{4 - s^2}$$

The identifications (42) give

$$\mathrm{OD}\ Z_a = \frac{s}{4 - s^2}$$

$$\mathrm{OD}\ Z_b = \frac{-s^3}{4 - s^2}$$

The p. r. impedance Z_a must be of the form

$$Z_a = \frac{a_2 s^2 + a_1 s + a_0}{s+2}$$

Furthermore, it must obey

$$2a_1 s - s(a_2 s^2 + a_0) = s: \sqrt{\pm[2(a_2 s^2 + a_0) - a_1 s^2}$$

Rational

The method for obtaining a rational z_{12} is essentially that of Kinariwala,[13] which is to solve the quadratic equation for the coefficients which will yield a perfect square, i. e.

$$c_2 s^4 \pm c_1 s^2 + c_0: \quad c_1 = \pm 2\sqrt{c_2 c_0} \tag{46}$$

For higher order functions, this problem is more readily solved by augmentation of the odd part. The solution of the present problem is

$$a_2 = a_0 = 0, \quad a_1 = \frac{1}{2}$$

Thus, the impedance Z_a is given by

$$Z_a = \frac{1/2\ s}{s+2}$$

Z_b must then be of the form

$$Z_b = \frac{b_2 s^2 + b_1 s + b_0}{s+2}$$

It is restricted by

$$2b_1 s - s(b_2 s^2 + b_0) = -s^3 : \sqrt{2(b_2 s^2 + b_0) - b_1 s^2}$$

Rational

The solution of these relations is given by

$$b_2 = 1, \quad b_1 = 2, \quad b_0 = 4$$

The impedance Z_b is

$$Z_b = \frac{s^2 + 2s + 4}{s + 2}$$

The even parts of these impedances are

$$EV\ Z_a = \frac{-1/2\ s^2}{4 - s^2}$$

$$EV\ Z_b = \frac{8}{4 - s^2}$$

The parameters which will realize the odd parts of the above impedances are

$$Z_a:\ z_{11a} = \frac{s}{4} \qquad\qquad Z_b:\ z_{11b} = s + \frac{4}{s}$$

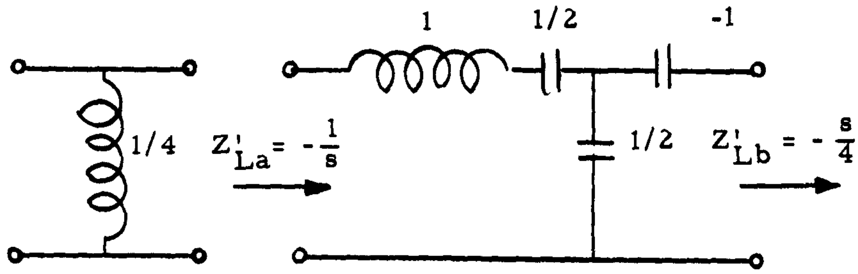
$$z_{12a} = \frac{s}{2\sqrt{2}} \qquad\qquad z_{12b} = \frac{2\sqrt{2}}{s}$$

$$z_{22a} = \frac{s}{2} \qquad\qquad z_{22b} = \frac{2}{s}$$

$$Z_{La} = \frac{2}{s} \qquad\qquad Z_{Lb} = \frac{s}{2}$$

These parameters are clearly those which may be realized in the cascade-feedback structure of Fig. 12. The procedure follows the later steps of section VII, and the LC realizations for the individual networks are given in Fig. 19. The final RC realization of Eq. (45) is given in Fig. 20.

This method will achieve the optimum--minimum pole sensitivity and minimum zero sensitivity. It is much more difficult computationally than other methods discussed, and in many practical situations may be abandoned for the minimum pole sensitivity method of section VII.



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a) LC Realization of

$$OD\ Z_a = \frac{s}{4 - s^2}$$

b) LC Realization of

$$OD\ Z_b = \frac{-s^3}{4 - s^2}$$

FIG. 19--THE COMPONENT NETWORKS FOR THE LC REALIZATION OF OD $Z = \frac{s - s^3}{4 - s^2}$

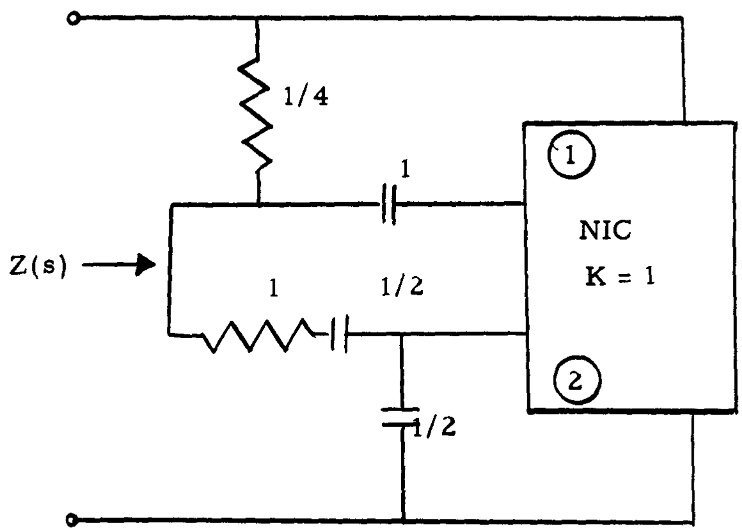

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FIG. 20--OPTIMALLY SENSITIVE REALIZATION OF $Z(s) = \frac{1-s}{4-s}$

IX. CONCLUSIONS

Methods have been presented for RC active driving-point synthesis which yield semi-unique structures from a straight-forward realization procedure. The networks obtained are usually practical (e. g., common-ground) and not given to an excess of elements. The limitations are few and usually may be circumvented by augmentation of the given function. A problem which warrants further consideration is, "Can any function, by suitable subtraction of constants (resistances) from it and its reciprocal, be made minimum active or minimum sensitivity (section VIII.) realizable?"

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